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Stabilization of the motions of mechanical systems with variable masses☆

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ABSTRACT

A holonomic mechanical system with variable masses and cyclic coordinates is considered. Such a system can have generalized steady motions in which the positional coordinates are constant and the cyclic velocities under the action of reactive forces vary according to a given law. Sufficient Routh-Rumyantsev-type conditions for the stability of such motions are determined. The problem of stabilizing a given translational-rotational motion of a symmetric satellite in which its centre of mass moves in a circular orbit and the satellite executes rotational motion about its axis of symmetry is solved.

The problem of the stability and stabilization of the steady motions of mechanical systems is of both great theoretical interest and considerable practical importance.[1–6] The same problem in the unsteady formulation has been investigated to a much lesser extent. Results are presented below which extend the results obtained in previous papers.[7–12]

1. Formulation of the problem

Consider a mechanical system with variable masses $m_\lambda = m_\lambda(t)$ $(\lambda = 1, 2, \ldots, N)$ under time-varying holonomic constraints such that the position of the system is determined by n independent generalized coordinates $q_1, q_2, \ldots, q_n$, and its kinetic energy has the form

$$T = T_2 + T_1 + T_0 = \frac{1}{2}\dot{\mathbf{q}}'\mathbf{A}(t, \mathbf{m}(t), \mathbf{q})\dot{\mathbf{q}} + \dot{\mathbf{q}}'\mathbf{B}(t, \mathbf{m}(t), \mathbf{q}) + C(t, \mathbf{m}(t), \mathbf{q})$$

$$\mathbf{q} = (q_1, q_2, \ldots, q_n)', \quad \mathbf{m} = (m_1, m_2, \ldots, m_N)'$$

The prime denotes transposition.

We shall assume that the masses of the points of the system $m_\lambda = m_\lambda(t)$ $(\lambda = 1, 2, \ldots, N)$ are bounded and do not vanish. Hence, in a non-degenerate system of coordinates, the component T_2 of the kinetic energy is a positive-definite quadratic form with respect to $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$.

The equations of motion of this system can be written in the form[13]

$$\frac{d^0}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{Q} + \mathbf{\Psi} \tag{1.1}$$

where d^0/dt is a derivative for fixed masses, $\mathbf{Q} = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}})$ is the resultant of the generalized active forces, and $\mathbf{\Psi} = \mathbf{\Psi}(t, \mathbf{q}, \dot{\mathbf{q}})$ is the resultant of the generalized reactive forces caused by the separation and joining of particles to material points of the system which are varying in mass and their motion within these points. When there is no relative motion of the particles within the material points of the system (the Meshcherskii case), the reactive force has the form[13,14]

$$\Psi_j = \sum_{k=1}^{N}\left(\frac{dm_{1k}}{dt}\bar{\mathbf{V}}_{1k}^2 - \frac{dm_{2k}}{dt}\bar{\mathbf{V}}_{2k}^2\right)\frac{\partial \bar{\mathbf{r}}_k}{\partial q_j}$$

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where dm_{1k}/dt and dm_{2k}/dt are the intensities of the joining and separation of the masses for the k-th point respectively, $\bar{\mathbf{V}}_{1k}^2$ and $\bar{\mathbf{V}}_{2k}^2$ are their relative velocities and $\bar{\mathbf{r}}_k$ is the radius vector of the k-th point.

We will assume that the kinetic energy of the system can be represented in the form

$$T = \frac{1}{2}\dot{\mathbf{r}}'\mathbf{A}_1\dot{\mathbf{r}} + \dot{\mathbf{r}}'\mathbf{B}_1 + g\left(\dot{\mathbf{r}}'\mathbf{A}_2\dot{\mathbf{s}} + \frac{1}{2}\dot{\mathbf{s}}'\mathbf{A}_3\dot{\mathbf{s}} + \dot{\mathbf{s}}'\mathbf{B}_2\right) + C \tag{1.2}$$

Here,

$$\mathbf{r} = (q_1, q_2, \dots, q_m)', \quad \mathbf{s} = (q_{m+1}, q_{m+2}, \dots, q_n)'$$

$$\mathbf{A}_1 = \mathbf{A}_1(t, \mathbf{m}(t), \mathbf{r}), \quad \mathbf{A}_1 \in R^{m\times m}, \quad \mathbf{B}_1 = \mathbf{B}_1(t, \mathbf{m}(t), \mathbf{r}), \quad \mathbf{B}_1 \in R^m$$

$$g = g(t) = g(\mathbf{m}(t)), \quad 0 < g_0 \le g(\mathbf{m}(t)) \le g_1$$

$$\mathbf{A}_2 = \mathbf{A}_2(t, \mathbf{r}) \in R^{m\times(n-m)}, \quad \mathbf{A}_3 = \mathbf{A}_3(t, \mathbf{r}) \in R^{(n-m)\times(n-m)}$$

$$\mathbf{B}_2 = \mathbf{B}_2(t, \mathbf{r}) \in R^{n-m}$$

The coefficient g depends solely on the varying masses of the points of the system, and the coefficients $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{B}_2$ are independent of these masses.

We shall also assume that the forces $\mathbf{Q}$ and $\mathbf{\Psi}$ are independent of the coordinate $\mathbf{s}$, where $\mathbf{Q} = (\mathbf{Q}^r, \mathbf{Q}^s)'$, $\mathbf{Q}^s \equiv 0$, $\mathbf{\Psi} = (\mathbf{\Psi}^r, \mathbf{\Psi}^s)'$, the coordinates $\mathbf{r}$ can be called positional coordinates and the coordinates $\mathbf{s}$ pseudocyclic coordinates.[6] In the case of system (1.2), Eqs (1.1) take the form

$$\frac{d^0}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial T}{\partial \mathbf{r}} = \mathbf{Q}^r + \mathbf{\Psi}^r, \quad \frac{d^0}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{s}}}\right) = \mathbf{\Psi}^s \tag{1.3}$$

We now introduce new impulse-type variables corresponding to the cyclic coordinates

$$\mathbf{p} = \mathbf{A}_2'\dot{\mathbf{r}} + \mathbf{A}_3\dot{\mathbf{s}} + \mathbf{B}_2 \tag{1.4}$$

defining the variables

$$\dot{\mathbf{s}} = \mathbf{A}_3^{-1}(\mathbf{p} - \mathbf{A}_2'\dot{\mathbf{r}} - \mathbf{B}_2) \tag{1.5}$$

in terms of them by virtue of the fact that $\det\mathbf{A}_3 \neq 0$.

We introduce the Routh function using the formula

$$R = T - \dot{\mathbf{s}}'\frac{\partial T}{\partial \dot{\mathbf{s}}}$$

substituting $\dot{\mathbf{s}}$ from equality (1.5). We have the following expression

$$R = R_2 + R_1 + R_0$$

$$R_2 = \frac{1}{2}\dot{\mathbf{r}}'(\mathbf{A}_1 - g\mathbf{A}_2\mathbf{A}_3^{-1}\mathbf{A}_2')\dot{\mathbf{r}}, \quad R_1 = g\mathbf{E}'\dot{\mathbf{r}}, \quad \mathbf{E} = (\mathbf{p} - \mathbf{B}_2)'\mathbf{A}_3^{-1}\mathbf{A}_2' + \mathbf{B}_1'$$

$$R_0 = C - \frac{1}{2}g(\mathbf{p} - \mathbf{B}_2)'\mathbf{A}_3^{-1}(\mathbf{p} - \mathbf{B}_2)$$

The equations of motion with respect to the positional coordinates (1.3) in terms of the Routh function are reduced to the form

$$\frac{d^0}{dt}\left(\frac{\partial R_2}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial R_2}{\partial \mathbf{r}} = \frac{\partial R_0}{\partial \mathbf{r}} + g\mathbf{G}\dot{\mathbf{r}} + \mathbf{D}^r$$

$$\mathbf{G} = \left(\frac{\partial \mathbf{E}}{\partial \mathbf{r}}\right)' - \frac{\partial \mathbf{E}}{\partial \mathbf{r}} = -\mathbf{G}' \tag{1.6}$$

$$\mathbf{D}^r = \mathbf{Q}^r + \mathbf{\Psi}^r - g\mathbf{A}_2\mathbf{A}_3^{-1}\mathbf{\Psi}^s - g\frac{\partial}{\partial t}(\mathbf{A}_2\mathbf{A}_3^{-1})\mathbf{p} + \frac{\partial^0}{\partial t}(g\mathbf{A}_2\mathbf{A}_3^{-1}\mathbf{B}_2 - \mathbf{B}_1) \tag{1.7}$$

We now suppose that the action of the active and reactive forces and the effect of the unsteady character of the constraints are such that the representation

$$\mathbf{D}^r = -\frac{\partial \Pi}{\partial \mathbf{r}} - \frac{\partial \Phi}{\partial \dot{\mathbf{r}}}$$

$$\Pi = \Pi(t, \mathbf{m}(t), \mathbf{p}, \mathbf{r}), \quad 2\Phi = \dot{\mathbf{r}}'\mathbf{F}(t, \mathbf{r})\dot{\mathbf{r}}, \quad \mathbf{F} \in R^{m\times m}, \quad \mathbf{F}' = \mathbf{F} \tag{1.8}$$

holds, where Π is a certain scalar function of the potential energy type and Φ is a function of the dissipative-accelerating type. With these assumptions, in accordance with relations (1.4) and (1.6)–(1.8), the equations of motion (1.3) reduce to the equations

$$\frac{d^0}{dt}\left(\frac{\partial R_2}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial R_2}{\partial \mathbf{r}} = -\frac{\partial W}{\partial \mathbf{r}} + g\mathbf{G}\dot{\mathbf{r}} - \frac{\partial \Phi}{\partial \dot{\mathbf{r}}}, \quad \frac{d\mathbf{p}}{dt} = \frac{1}{g}\mathbf{\Psi}^s$$
$$\frac{d\mathbf{s}}{dt} = \dot{\mathbf{s}}, \quad \dot{\mathbf{s}} = \mathbf{A}_3^{-1}(\mathbf{p} - \mathbf{B}_2 - \mathbf{A}_2'\dot{\mathbf{r}}) \tag{1.9}$$

where $W(t, \mathbf{m}(t), \mathbf{r}, \mathbf{c}) = \Pi - R_0$ is a scalar function of the reduced potential energy type.[6]

We shall assume that the functions appearing in Eqs (1.9) are bounded and continuous, and that they have bounded continuous derivatives with respect to $(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{p})$ in the domain $R^+ \times \Gamma_0$, where

$$\Gamma_0 = \{\|\dot{\mathbf{r}}\| \le H_0 > 0, \|\mathbf{r}\| \le H_0, \|\mathbf{p}\| \le \rho_0 > 0\}, \quad \|\mathbf{r}\|^2 = \sum_{j=1}^{m} r_j^2$$

It therefore follows that Eqs (1.9) are precompact and that a family of analogous limiting systems can be matched to them.[7–9]

Suppose the second set of equations (1.9), when $\mathbf{\Psi}^s = \mathbf{\Psi}^s(t) = g(\mathbf{m}(t))\mathbf{p}_0(t)$, has a bounded solution

$$\mathbf{p} = \mathbf{p}_0(t), \quad \|\mathbf{p}_0(t)\| \le \rho_1 < \rho_0$$

Here, $\partial W(t, \mathbf{m}(t), \mathbf{r}, \mathbf{p}_0(t))/\partial \mathbf{r} = 0$, if $\mathbf{r} = 0$. System (1.9) will then have a generalized steady motion

$$\dot{\mathbf{r}} = \mathbf{r} = \mathbf{0}, \quad \dot{\mathbf{s}} = \dot{\mathbf{s}}_0(t) = \mathbf{A}_3^{-1}(t, \mathbf{0})(\mathbf{p}_0(t) - \mathbf{B}_2(t, \mathbf{0})), \quad \mathbf{s}(t) = \int_{t_0}^{t} \dot{\mathbf{s}}_0(\tau)d\tau + \mathbf{s}_0 \tag{1.10}$$

in which the cyclic velocities $\dot{\mathbf{s}}_0(t)$ are variable functions in the general case.

2. A theorem on stabilization

We will now consider the problem of stabilizing the generalized steady motion (1.10) by the action of active and reactive forces that are reducible to the form (1.8).

For convenience, a bounded, continuous function such that a number $T > 0$ is found for each $L > 0$ and for which the inequality

$$\int_{t_0}^{t_0+T} \alpha(\tau)d\tau \le -L \tag{2.1}$$

is satisfied for any $t_0 \ge 0$ will be denoted by $\alpha: R^+ \to R$.

Theorem 1. *We will assume that*:

1) *the function*

$$W_1(t, \mathbf{m}(t), \mathbf{r}) = W(t, \mathbf{m}(t), \mathbf{r}, \mathbf{p}_0(t)) - W(t, \mathbf{m}(t), \mathbf{0}, \mathbf{p}_0(t))$$

is definite-positive and allows of an infinitesimal higher bound with respect to $\mathbf{r}$;

2) *the generalized steady motion* (1.10) *is isolated when* $\mathbf{p} = \mathbf{p}_0(t)$ *in such a manner that*

$$\left\|\frac{\partial W}{\partial \mathbf{q}}(t, \mathbf{m}(t), \mathbf{r}, \mathbf{p}_0(t))\right\| \ge \delta = \delta(\varepsilon) > 0, \quad \forall t \ge 0, \ \forall \mathbf{r} \in \{0 < \varepsilon \le \|\mathbf{r}\| \le H_0\} \tag{2.2}$$

3) *the action of the active and reactive forces is such that the inequalities*

$$\sum_{\lambda=1}^{N} \frac{\partial R_2}{\partial m_\lambda}\dot{m}_\lambda - 2\Phi \le \mu_1(t)R_2, \quad \frac{\partial W_1}{\partial t} + \sum_{\lambda=1}^{N} \frac{\partial W_1}{\partial m_\lambda}\dot{m}_\lambda \le \mu_2(t)W_1 \tag{2.3}$$

$$\int_0^\infty \mu(t)dt \le M = \text{const}, \quad \mu(t) = \max(\mu_1(t), \mu_2(t)) \tag{2.4}$$

are satisfied for certain bounded continuous functions μ_1, μ_2: $R^+ \to R$ *for all* $(t, \mathbf{q}, \dot{\mathbf{q}}) \in R^+ \times \Gamma_0$ *and, at the same time, the inequality* $\mu_1^*(t) \not\equiv \mu_2^*(t)$ *holds for any functions* $\mu_1^*(t)$ *and* $\mu_2^*(t)$ *which pass in the limit to* $\mu_1(t)$ *and* $\mu_2(t)$.

Then, when $\mathbf{\Psi}^s \equiv \mathbf{\Psi}_0^s(t)$, the generalized steady motion (1.10) is uniformly stable and uniformly asymptotically stable with respect to $(\dot{\mathbf{r}}, \mathbf{r})$ with respect to motions for which $\mathbf{p} \equiv \mathbf{p}_0(t)$. If, however, when $\mathbf{\Psi}^s \not\equiv \mathbf{\Psi}_0^s(t)$,

$$(\mathbf{\Psi}^s - \mathbf{\Psi}_0^s(t))'(\mathbf{p} - \mathbf{p}_0(t)) \le \alpha(t)\|\mathbf{p} - \mathbf{p}_0(t)\|^2 \tag{2.5}$$

then the steady motion (1.10) is uniformly asymptotically stable with respect to $(\dot{\mathbf{r}}, \mathbf{r}, \mathbf{p} - \mathbf{p}_0(t))$.

Proof. We will use the stability theorem in Ref. 10. For the function $V_1 = ||\mathbf{p} - \mathbf{p}^0(t)||^2$, we find $\dot{V}_1 \equiv 0$ or, in the case of condition (2.5),

$$\dot{V}_1 \le 2\alpha(t)\|\mathbf{p} - \mathbf{p}_0(t)\|^2 = 2\alpha(t)V_1$$

According to inequality (2.1), it is sufficient for deriving of the result[10] that the uniform asymptotic stability of the motion (1.10) with respect to motions for which $V_1 \equiv 0$ or $\mathbf{p} \equiv \mathbf{p}_0(t)$ should hold. This is established on the basis of the function $V_2 = R_2 + W_1$ that is positive-definite with respect to $(\dot{\mathbf{r}}, \mathbf{r})$. By virtue of Eqs (1.9), for its derivative we find from condition (2.3)

$$\begin{aligned}\dot{V}_2 &= \frac{\partial W_1}{\partial t} + \sum_{\lambda=1}^{N} \frac{\partial (R_2 + W_1)}{\partial m_\lambda}\dot{m}_\lambda - 2\Phi \le \mu_1(t)R_2 + \mu_2(t)W_1 = \\ &= \mu(t)V_2 - V_3 \le \mu(t)V_2 \\ V_3 &= (\mu(t) - \mu_1(t))R_2 - (\mu(t) - \mu_2(t))W_1 \ge 0\end{aligned}$$

It follows from inequality (2.4) that the solution $u = 0$ of the comparison equation $\dot{u} = \mu(t)u$ is uniformly stable. The function V_3^* which passes in the limit to V_3 is analogous and, by virtue of the inequality $\mu_1^* \not\equiv \mu_2^*$, the set

$$\{V_3^* = 0\} = \{(\mu^*(t) - \mu_1^*(t))R_2^* + (\mu^*(t) - \mu_2^*(t))W^* = 0\}$$

is $\{R_2^* = 0, \mu^*(t) = \mu_2^*(t)\}$ or $\{W^* = 0, \mu^*(t) = \mu_1^*(t)\}$.

According to condition 1) of the theorem, it does not contain motions corresponding to the value $\mathbf{p} = \mathbf{p}_0(t)$, apart from $\dot{\mathbf{r}} = \mathbf{r} = \mathbf{0}$. Using well-known theorems (Ref. 10, and Theorem 3.4), we obtain the required result.

Remark. The condition $\mu_1^*(t) \not\equiv \mu_2^*(t)$ is necessarily satisfied if, for a certain $\delta_0 > 0$, a sequence

$$t_n \to +\infty, \quad 0 < t_0 \le t_{n+1} - t_n \le T$$

exists such that $||\mu_2(t)| - |\mu_1(t)|| \ge \delta_0$ for $t \in [t_n, t_n + t_0]$.

3. The problem of stabilizing the motion of a satellite

We will now consider the problem of stabilizing the translational-rotational motion of a satellite in a gravitational field under the action of reactive forces, assuming that the equations of motion of the centre of mass are separated from the equations of motion of the satellite with respect to the centre of mass.

We will take the cylindrical coordinates r, φ, z as the variables describing the motion of the centre of mass.[14] The equations of motion in the variables $r, \upsilon_\tau = r\dot{\varphi}, z$ have the form

$$\begin{aligned}\ddot{r} &= -\frac{\mu r}{\Delta(r,z)} + \frac{\upsilon_\tau^2}{r} + \frac{1}{m(t)}\Psi^r \\ \dot{\upsilon}_\tau &= -\frac{\dot{r}}{r}\upsilon_\tau + \frac{1}{m(t)}\Psi^\tau, \quad \ddot{z} = -\frac{\mu z}{\Delta(r,z)} + \frac{1}{m(t)}\Psi^z \\ \Delta(r,z) &= (r^2 + z^2)^{3/2}\end{aligned} \tag{3.1}$$

where μ is the gravitational constant, $m = m(t)$ is the variable mass of the satellite and $\overline{\mathbf{\Psi}} = (\Psi^r, \Psi^\tau, \Psi^z)'$ is the vector of the reactive forces. When $\overline{\mathbf{\Psi}} = 0$, Eq. (3.1) has the solution

$$r = r_0 = \text{const} > 0, \quad \upsilon_\tau = \upsilon_{\tau 0} = r_0 \quad \dot{\varphi}_0 = \text{const}, \quad z = 0, \quad r_0\upsilon_{\tau 0}^2 = \mu \tag{3.2}$$

which corresponds to the motion of the centre of mass of the satellite in a circular orbit.

We now set up the equations of the perturbed motion with Eqs (3.1), assuming that $x = r - r_0$, $y = \upsilon_\tau - \upsilon_{\tau 0}$. We obtain

$$\begin{aligned}\ddot{x} &= -\frac{\mu(r_0 + x)}{\Delta(r_0 + x, z)} + \frac{\mu}{r_0(r_0 + x)} + \frac{2\upsilon_{\tau 0}y + y^2}{r_0 + x} + \frac{1}{m}\Psi^r \\ \dot{y} &= -\frac{\dot{x}}{r_0 + x}(\upsilon_{\tau 0} + y) + \frac{1}{m}\Psi^\tau, \quad \ddot{z} = -\frac{\mu z}{\Delta(r_0 + x, z)} + \frac{1}{m}\Psi^z\end{aligned} \tag{3.3}$$

We will assume that the reactive forces, as stabilizing forces, are defined by the relations

$$\frac{1}{m}\Psi^r = -\beta_1(t)x - \beta_2(t)\dot{x}, \quad \frac{1}{m}\Psi^\tau = -\beta_3(t)y, \quad \Psi^z = 0$$

$$\beta_1(t) \ge \frac{\mu}{r_0^2} + \delta, \quad \delta > 0, \quad \dot{\beta}_1(t) \le 0, \quad \beta_2(t) \ge 0, \quad \beta_3(t) \ge 0 \tag{3.4}$$

The Lyapunov function

$$V = \frac{1}{2}(\dot{x}^2 + y^2 + \dot{z}^2) + \upsilon_{\tau0}y - \upsilon_{\tau0}^2\ln\left(1 + \frac{y}{\upsilon_{\tau0}}\right) - \frac{\mu}{\sqrt{(r_0+x)^2 + z^2}} +$$

$$+\frac{\mu}{r_0} - \frac{\mu}{r_0}\ln\left(1 + \frac{x}{r_0}\right) + \beta_1(t)\frac{x^2}{2} \tag{3.5}$$

is positive-definite with respect to $\dot{x}, \dot{z}, y, x, z$. By virtue of Eqs (3.3), its derivative is

$$\dot{V} \le -\beta_2(t)\dot{x}^2 - \beta_3(t)y^2 \le 0$$

Hence, the steady motion of the satellite (3.2), under the action of reactive forces of the form (3.4), is stable with respect to $\dot{x}, \dot{z}, y, x, z$. It may be found that the set $\{y = 0\}$ does not contain the motions of system (3.3) except for the motion (3.2) and the set $\{\dot{x} = 0\}$ only contains the motions

$$\{x(t) = x_0 = \text{const},\ y(t) = y_0 = \text{const},\ z(t) = 0,\ (r_0 + x_0)(\upsilon_{\tau0} + y_0)^2 = \mu\}$$

that is, the motions of the satellite in a circular orbit corresponding to the perturbed motions $\upsilon_{\tau0} + y_0$. Then, applying a well-known theorem,[15] we find that, in the case of the condition $\beta_3(t) \ge \beta_0 > 0$ in the sequence of intervals

$$[t_n, t_n + s], \quad t_n \to +\infty, \quad 0 < s < t_{n+1} - t_n \le \rho$$

the motion (3.2) is uniformly asymptotically stable and, if $\beta_3(t) = 0$, $\beta_2(t) \ge \beta_0 > 0$, then the motion (3.2) is asymptotically stable and each perturbed motion of the satellite converges indefinitely to motion in a circular orbit in the $z = 0$ plane.

At the same time, the explicit expression (3.5) for the Lyapunov function enables us to determine the attraction domain.

Hence, we shall assume that the centre of mass O of the satellite moves uniformly ($\omega_0 = \dot{\varphi}_0 = \text{const}$) in a circular orbit. We will now consider the motion of a symmetric satellite about its centre of mass under the action of the moment of the gravitational forces[3,14,16] and the reactive moment. We will assume that the change in the mass of the satellite does not lead to a change in the directions of the principal central axes of inertia Ox_1, Ox_2, Ox_3 with moments of inertia

$$A(t) = B(t), \quad C(t), \quad A(t) \ge A_0 > 0, \quad C(t) \ge C_0 > 0$$

The rotational motion of the satellite with respect to the orbital system of coordinates, in which the Oz axis is directed along the radius vector of the centre of mass, the Ox axis is directed along the normal to the orbital plane and the Oy axis is directed along the transversals, is determined by the Eulerian angles ψ, θ and φ which are introduced in the usual manner.[3,14]

The kinetic energy of the rotational motion of the satellite and the potential energy of the Newtonian forces are defined by the equalities

$$T = \tilde{T} + \frac{1}{2}C(t)(\dot{\varphi} + \dot{\psi}\cos\theta - \omega_0\sin\theta\cos\psi)^2, \quad \Pi = \frac{3}{2}(C - A)\omega_0^2\cos^2\theta$$

$$\tilde{T} = \frac{1}{2}A(t)(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta + 2\omega_0(\dot{\theta}\sin\psi + \dot{\psi}\sin\theta\cos\theta\cos\psi) - \omega_0^2\sin^2\theta\cos^2\psi)$$

We will now show that the moments of the reactive forces in projections on to the axes of system $Ox_1x_2x_3$ in the form

$$M_1 = -k(t)\bar{\omega}_1, \quad M_2 = -k(t)\bar{\omega}_2, \quad M_3 = M_3(t, \omega_3) \tag{3.6}$$

where $\bar{\omega}_1$ and $\bar{\omega}_2$ are the projections of the relative velocity of the satellite on to the Ox_1 and Ox_2 axes and ω_3 is the projection of the absolute angular velocity on to the Ox_3 axis, ensure stabilization of the motion in which the satellite executes a specified unsteady rotation about the Ox_3 axis, which maintains an unchanged position in the $Oxyz$ system.

Correspondingly, for the generalized forces, we have

$$Q_\theta = -k(t)\dot{\theta}, \quad Q_\psi = -k(t)\dot{\psi}\sin^2\theta + M_3\cos\theta, \quad Q_\varphi = M_3$$

The equation of motion of the satellite with respect to the coordinate φ has the form

$$C(t)\frac{dp}{dt} = M_3, \quad p = \omega_3 = \dot{\varphi} + \dot{\psi}\cos\theta - \omega_0\sin\theta\cos\psi$$

We now define the Routh function

$$R = \tilde{T} - \frac{1}{2}C(t)p^2 + C(t)p(\dot{\psi}\cos\theta - \omega_0\sin\theta\cos\psi)$$

By carrying out the necessary calculations, it can be found that the equations of motion of the satellite reduce to the form (1.9) with the functions

$$R_2 = \frac{1}{2}A(t)(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta)$$

$$W(t, \theta, \psi) = \frac{1}{2}A(t)\omega_0^2\sin^2\theta\cos^2\psi + \frac{3}{2}\omega_0^2(C(t) - A(t))\cos^2\theta + C(t)p\omega_0\sin\theta\cos\psi$$

$$\Phi(t, \dot{\psi}, \dot{\theta}, \theta) = \frac{1}{2}k(t)(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2)$$

From the equations

$$\frac{\partial W}{\partial\theta} = (A(t)\cos^2\psi - 3C(t) + 3A(t))\omega_0^2\sin\theta\cos\theta + C(t)p_0(t)\cos\theta\cos\psi = 0$$
$$\frac{\partial W}{\partial\psi} = -A(t)\omega_0^2\sin^2\theta\sin\psi\cos\psi - C(t)p_0(t)\sin\theta\sin\psi = 0 \tag{3.7}$$

we find that, when a moment $M_3^0(t) = C(t)\dot{\omega}_3^0(t)$ acts, the satellite can execute the specified generalized steady motion

$$\dot{\theta} = 0, \quad \theta_0 = \pi/2, \quad \dot{\psi} = 0, \quad \psi_0 = \pi, \quad \omega_3 = p_0(t) = \omega_3^0(t) \tag{3.8}$$

Besides this motion, in the case of this moment there is a whole family of generalized steady motions differing from (3.8) by the constant term $\omega_3 = \omega_3^0(t) + c$.

For the motion (3.8), the function

$$W_1(t, \theta, \psi, p_0(t)) = W(t, \theta, \psi, p_0(t)) - W\left(t, \frac{\pi}{2}, \pi, p_0(t)\right) =$$
$$= \frac{3}{2}(C(t) - A(t))\omega_0^2\cos^2\theta - \frac{1}{2}A\omega_0^2(1 - \sin^2\theta\sin^2\psi) + C(t)p_0(t)\omega_0(1 + \sin\theta\cos\psi)$$

is positive-definite with respect to $x = (\theta - \pi/2)$, $y = (\psi - \pi)$ if the conditions

$$\nu_1(t) = C(t)\omega_3(t)\omega_0 + (3C(t) - 4A(t))\omega_0^2 \geq \delta_1 = \text{const} > 0$$
$$\nu_2(t) = C(t)\omega_3(t)\omega_0 - A(t)\omega_0^2 \geq \delta_1 \tag{3.9}$$

are satisfied. For the same conditions, if $|x| + |y| \geq \varepsilon > 0$, then

$$\left|\frac{\partial W_1}{\partial\theta}\right| \geq \delta_2(\varepsilon) > 0, \quad \left|\frac{\partial W_1}{\partial\psi}\right| \geq \delta_2 \tag{3.10}$$

Using Theorem 1, we find that the moments (3.6) when $M_3 = M_3^0(t)$ ensure asymptotic stability with respect to $\dot{\theta}$, $\dot{\psi}$, x, y and stability with respect to $\dot{\varphi}$ of the specified rotational motion of the satellite for any changes in its parameters $A(t)$, $C(t)$, $k(t)$ which satisfy inequalities (3.9) and the relations

$$\dot{A}(t) \leq 2k(t) + \mu_1(t)A(t), \quad \dot{\nu}_1(t) \leq \mu_2(t)\nu_1(t), \quad \dot{\nu}_2(t) \leq \mu_2(t)\nu_2(t)$$
$$\int_0^\infty \max(\mu_1(t), \mu_2(t))dt \leq M = \text{const}, \quad \left||\mu_1(t)| - |\mu_2(t)|\right| \geq \delta_3 > 0 \tag{3.11}$$

For these same conditions, the moments (3.6) with the value

$$M_3(t) = \alpha(t)(\omega_3 - \omega_3^0(t)) + C(t)\dot{\omega}_3^0(t)$$

ensure the total uniform asymptotic stability of the motion (3.8).

When $M_3(t) \equiv 0$ and $\omega_3(t) = \mathbf{p}_0(t) \equiv 0$, Eqs (3.7) have the solution

$$\theta = \pi/2, \quad \psi = \pi/2$$

A relative equilibrium position of the satellite with the axis of symmetry Ox_3 directed along an orbit that is tangential to the plane:

$$\dot{\theta} = \dot{\psi} = 0, \quad \theta = \pi/2, \quad \psi = \pi/2, \quad \omega_3 = 0 \tag{3.12}$$

corresponds to this. The function $W(t, \theta, \psi)$ when $\mathbf{p}_0(t) = 0$ is positive-definite with respect to $x = (\theta - \pi/2)$ and $y = (\psi - \pi/2)$ if the condition

$$\gamma_1(t) = (C(t) - A(t))\omega_0^2 \geq \delta_2 > 0$$

is satisfied. With the same condition, when account is taken of the inequality $A(t) \geq A_0 > 0$, conditions (3.10) are satisfied.

Suppose the changes in the parameters $A(t)$, $C(t)$, $k(t)$ satisfy the inequalities

$$\dot{A}(t) \leq 2k(t) + \mu_1(t)A(t), \quad \dot{A}(t) \leq \mu_2(t)A(t), \quad \dot{\gamma}_1(t) \leq \mu_2(t)\gamma_1(t)$$

Then, applying Theorem 1, we obtain that the moments (3.6) when $M_3 \equiv 0$ ensure uniform stability of the motion (3.12) with respect to $\dot{\theta}$, $\dot{\psi}$, x, y, ω_3 and, at the same time, each perturbed motion in which $\omega_3(t) \equiv 0$, indefinitely approaches the equilibrium position (3.12) when $t \to +\infty$. If $M_3 = \alpha(t)\omega_3$, the motion (3.12) is uniformly asymptotically stable.

When $M_3 \equiv 0$ and, correspondingly, $p = 0$, the satellite has a position of relative equilibrium in which its axis of symmetry is directed onto the attracting centre. As previously, it may be found that, in the case of conditions relating to the parameters $A(t)$, $C(t)$, $k(t)$ in the form of the inequalities

$$\gamma_2(t) = (A(t) - C(t))\omega_0^2 \geq \delta_2 > 0$$

$$\dot{A}(t) \leq 2k(t) + \mu_1(t)A(t), \quad \dot{A}(t) \leq \mu_2(t)A(t), \quad \dot{\gamma}_2(t) \leq \mu_2(t)\gamma_2(t)$$

the moment (3.6), when $M_3 = \alpha(t)\omega_3$, ensures the uniform asymptotic stability of this equilibrium position.

According to the technique of using sign-definite Lyapunov functions in stability and stabilization problems presented earlier,[10,12] the complete stabilizability of the translational-rotational motion of a satellite follows from the uniform asymptotic stability of the motion of the centre of mass of the satellite in a circular orbit and the uniform asymptotic stability of the rotational motion of the satellite about its centre of mass with the assumption that it moves in a circular orbit. It then follows that the reactive forces

$$\Psi_x = -\beta_1(t)x, \quad \Psi_y = -\beta_3(t)y, \quad \Psi_z = 0$$

forming the moment with respect to the centre of mass

$$M_1 = -k(t)\bar{p}, \quad M_2 = -k(t)\bar{q}, \quad M_3 = \alpha_1(t)(\omega_3 - \omega_3^0(t)) + C\dot{\omega}_3^0(t)$$

ensure, in the case of conditions (3.9) and (3.11), total stabilization of the translational-rotational motion of the satellite

$$r = r_0, \quad \upsilon_\tau = \upsilon_{\tau 0}, \quad z = 0, \quad \dot{\theta} = 0, \quad \dot{\psi} = 0, \quad \theta = \pi/2, \quad \psi = \pi, \quad \omega_3 = \omega_3^0(t)$$

in which the centre of mass of the satellite moves in a circular orbit and the satellite rotates with a specified angular velocity about an axis of symmetry which is perpendicular to the orbital plane. Analogous assertions also hold in the problem of stabilizing the motion in which the axis of symmetry of the satellite is directed along the radius vector of the centre of mass and along a tangent to a circular orbit.

Note that a number of the results obtained in this paper are also new for systems with constant masses.

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